(8) The Addition of Angular Momenta

① Tensor products and direct sums
[$\otimes$] [$\oplus$]

- They are to combine two Hilbert spaces into one.
  (vector)

a. tensor products.

- For a vector $\vec{v} \in V$ and the other $\vec{w} \in W$

$$\longrightarrow \quad \vec{v} \otimes \vec{w} \in V \otimes W .$$

ex. a system of two spin-$\frac{1}{2}$ particles.

the state ket of the system:

$$|\Psi\rangle = C_{\uparrow\uparrow} |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 + C_{\uparrow\downarrow} |\uparrow\rangle_1 \otimes |\downarrow\rangle_2$$
$$+ C_{\downarrow\uparrow} |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 + C_{\downarrow\downarrow} |\downarrow\rangle_1 \otimes |\downarrow\rangle_2$$

- Addition of two operators that are in ~~different~~
  H-spaces.

$$O_v \in V , \quad O_w \in W$$

$$\Rightarrow \quad O_v + O_w \equiv O_v \otimes I_w + I_v \otimes O_w$$

- Operator - ket multiplication
  (matrix) (vector)

$$\rightarrow (S_1 + S_2) |\Psi\rangle = C_{\uparrow\uparrow} \left[(S_1 |\uparrow\rangle_1) \otimes |\uparrow\rangle_2 + |\uparrow\rangle_1 \otimes (S_2 |\uparrow\rangle_2) \right]$$
$$+ \cdots$$

Rule ↑↑↑

$$(O_v \otimes I_w + I_v \otimes O_w) \cdot (|v\rangle \otimes |w\rangle)$$
$$= \left(O_v |v\rangle \right) \otimes \left(I_w |w\rangle \right) + \left(I_v |v\rangle \right) \otimes \left(O_w |w\rangle \right)$$

- base kets and dimension of the combined space.

$$V\ \{|v_i\rangle\} \quad \otimes \quad W\ \{|w_j\rangle\} \quad \rightarrow \quad V\otimes W\ \{|v_i\rangle \otimes |w_j\rangle\}$$

$\dim[V]$ $\quad$ $\dim[W]$ $\quad$ $\dim[V\otimes W]$

$= \dim[V]\cdot\dim[W]$!

b. Direct sums.

$$V\ \{|v_i\rangle\} \quad \oplus \quad W\ \{|w_j\rangle\} \quad \rightarrow \quad V\oplus W\ \{|v_i\rangle \cdots |w_j\rangle\}$$

$\dim[V]$ $\quad$ $\dim[W]$ $\quad$ $\dim[V\oplus W]$

$= \dim[V] + \dim[W]$.

ex. matrix representation of $\mathcal{D}(R)$

$\Rightarrow \mathcal{D}^{(j)}_{m'm}(R)$

$$\begin{pmatrix} \boxed{j=1} & & \\ & \boxed{j=2} & \\ & & \ddots \end{pmatrix} \Leftarrow V_{j=1} \oplus V_{j=2} \oplus \cdots$$

"Block - diagonal".

We have already seen the tensor product many times.

$$\to \quad |x, y, z\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle$$

$$H = \frac{\vec{P}^2}{2m} = \frac{1}{2m}\left((\tilde{P}_x \otimes I_y \otimes I_z)^2 + (I_x \otimes \tilde{P}_y \otimes I_z)^2 + (I_x \otimes I_y \otimes \tilde{P}_z)^2 \right)$$

c. notations.

1. DO NOT mix the index ordering.

ex. $\cancel{|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + |\uparrow\rangle_2 \otimes |\downarrow\rangle_1}$ (X)

$|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + |\downarrow\rangle_1 \otimes |\uparrow\rangle_2$ (O)

2. $\otimes$ (index) are often omitted for brevity.

ex. $|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \equiv |\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle$

: assuming that index ordering (1, 2) is fixed.

② Simple examples of Angular Momentum addition

- base ket of a spin-$\frac{1}{2}$ particle in space.

$$|x;\uparrow\rangle \equiv |x\rangle \otimes |\uparrow\rangle, \quad |x;\downarrow\rangle \equiv |x\rangle \otimes |\downarrow\rangle$$

- Wave function :

$$\langle x;\uparrow|\alpha\rangle \equiv \psi_\uparrow(x), \quad \langle x;\downarrow|\alpha\rangle \equiv \psi_\downarrow(x).$$

$$\text{or} \quad \Psi(x) \equiv \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix}.$$

- L·S coupling (spin-orbit interaction)

$$\vec{J} = \vec{L} + \vec{S} \quad (\vec{L}\cdot\vec{S} = \vec{J}^2 - \vec{L}^2 - \vec{S}^2)$$

$$= \vec{L} \otimes I + I \otimes \vec{S} \quad \text{also.} \quad \mathcal{D}(R) = \underset{(\vec{L})}{\mathcal{D}^{(orb)}}(R) \otimes \underset{(\vec{S})}{\mathcal{D}^{(spin)}}(R).$$

- Addition of $\ell=1$, $s=\frac{1}{2}$: $\vec{J}=\vec{L}+\vec{S}$

  - base vectors (kets)

$$|s,m_s\rangle : \ |\tfrac12,\tfrac12\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}, \quad |\tfrac12,-\tfrac12\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$$

$$|\ell,m_\ell\rangle : \ |1,1\rangle \doteq \begin{pmatrix}1\\0\\0\end{pmatrix}, \ |1,0\rangle \doteq \begin{pmatrix}0\\1\\0\end{pmatrix}, \ |1,-1\rangle \doteq \begin{pmatrix}0\\0\\1\end{pmatrix}$$

  - Angular momentum operators

$$s=\tfrac12 : \ S_+ \doteq \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}, \ S_- \doteq \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}, \ S_z \doteq \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\ell=1 : \ L_+ \doteq \hbar\begin{pmatrix}0&\sqrt2&0\\0&0&\sqrt2\\0&0&0\end{pmatrix}, \ L_- = \hbar\begin{pmatrix}0&0&0\\\sqrt2&0&0\\0&\sqrt2&0\end{pmatrix}, \ L_z \doteq \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$\downarrow$ in the form of the tensor product $(\ell=1)\otimes(s=\frac12)$

$$L_+\otimes I = \hbar \begin{array}{c} \begin{array}{ccc} m_\ell=1 & m_\ell=0 \ \left(m_s=\tfrac12 \ \vdots \ m_s=-\tfrac12\right) & m_\ell=-1 \end{array} \\ \left(\begin{array}{c|c|c} 0 & \begin{smallmatrix}\sqrt2&0\\0&\sqrt2\end{smallmatrix} & 0 \\ \hline 0 & 0 & \begin{smallmatrix}\sqrt2&0\\0&\sqrt2\end{smallmatrix} \\ \hline 0 & 0 & 0 \end{array}\right) \end{array}, \quad L_-\otimes I = \hbar\left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \begin{smallmatrix}\sqrt2&0\\0&\sqrt2\end{smallmatrix} & 0 & 0 \\ \hline 0 & \begin{smallmatrix}\sqrt2&0\\0&\sqrt2\end{smallmatrix} & 0 \end{array}\right)$$

$$I\otimes S_+ = \hbar\left(\begin{array}{c|c|c} \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & 0 & 0 \\ \hline 0 & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & 0 \\ \hline 0 & 0 & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} \end{array}\right), \quad I\otimes S_- = \hbar\left(\begin{array}{c|c|c} \begin{smallmatrix}0&0\\1&0\end{smallmatrix} & 0 & 0 \\ \hline 0 & \begin{smallmatrix}0&0\\1&0\end{smallmatrix} & 0 \\ \hline 0 & 0 & \begin{smallmatrix}0&0\\1&0\end{smallmatrix} \end{array}\right)$$

$$L_z\otimes I = \hbar\left(\begin{array}{cc|cc|cc} 1 & & & & & \\ & 1 & & & & \\ \hline & & 0 & & & \\ & & & 0 & & \\ \hline & & & & -1 & \\ & & & & & -1 \end{array}\right), \quad I\otimes S_z = \frac{\hbar}{2}\left(\begin{array}{cc|cc|cc} 1 & & & & & \\ & -1 & & & & \\ \hline & & 1 & & & \\ & & & -1 & & \\ \hline & & & & 1 & \\ & & & & & -1 \end{array}\right)$$

$$J_+ = L_+ \otimes I + I \otimes S_+ \quad , \quad J_- = L_- \otimes I + I \otimes S_-$$

$$= \hbar \begin{pmatrix} 0 & 1 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 1 & 0 \end{pmatrix}$$

$$J_z = L_z \otimes I + I \otimes S_z$$

$$= \hbar \begin{pmatrix} \frac{3}{2} & & & & & \\ & \frac{1}{2} & & & & \\ & & \frac{1}{2} & & & \\ & & & -\frac{1}{2} & & \\ & & & & -\frac{1}{2} & \\ & & & & & -\frac{3}{2} \end{pmatrix}$$

$\rightarrow$ see if

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$

holds. Yes!

eigenkets.

$|j, m\rangle \quad \rightarrow$

It's obvious that

$$\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle \doteq \begin{array}{cl} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{array}{l} m_l = 1,\ m_s = \frac{1}{2} \\ m_s = -\frac{1}{2} \\ m_l = 0 \\ \\ m_l = -1 \\ \\ \end{array} \end{array} , \qquad \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle \doteq \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

* check if
$$\begin{cases} J_+ \left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle = 0 \\ J_- \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = 0 \end{cases}$$

Lowering with J_- :

$$J_- \left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle = \hbar \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \hbar\sqrt{3} \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle \quad \text{---- } (*)$$

$\uparrow$ From $\sqrt{(j+m)(j-m+1)}\, \hbar\, |j, m-1\rangle = J_- |j, m\rangle$

Raising with J_+ :

$$J_+ \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = \hbar \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ 1 \\ 0 \end{pmatrix} = \hbar\sqrt{3} \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle \quad \text{--- } (**)$$

$\uparrow$ From $\sqrt{(j-m)(j+m+1)}\, \hbar\, |j, m+1\rangle = J_+ |j, m\rangle$

also, by just using operators and kets.

$$J_-\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle = (L_-\otimes I + I\otimes S_-)\,(\overset{l\ \ m_l}{|1,1\rangle}\otimes\overset{s\ \ m_s}{|\tfrac{1}{2},\tfrac{1}{2}\rangle})$$

$$= \hbar\sqrt{2}\,|1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle + \hbar\,|1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\sqrt{3}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle$$

$$\Rightarrow \left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\doteq \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$J_+\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle = (L_+\otimes I + I\otimes S_+)\,(|1,-1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle)$$

$$= \hbar\,|1,-1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle + \hbar\sqrt{2}\,|1,0\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\sqrt{3}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

$$\Rightarrow \left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\,|1,-1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ 1 \\ 0 \end{pmatrix}$$

by using the orthogonality $\oplus$ "convention",

we may write

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = -\sqrt{\tfrac{1}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = -\sqrt{\tfrac{2}{3}}\,|1,-1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$\Rightarrow$ "Clebsch-Gordan Coefficients" ☆☆☆

$$\begin{pmatrix} |\tfrac{3}{2},\tfrac{1}{2}\rangle \\ |\tfrac{1}{2},+\tfrac{1}{2}\rangle \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{\tfrac{2}{3}} & \sqrt{\tfrac{1}{3}} \\ -\sqrt{\tfrac{1}{3}} & \sqrt{\tfrac{2}{3}} \end{pmatrix}}_{\text{orthogonal matrix}} \begin{pmatrix} |1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle \\ |1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle \end{pmatrix}$$

So, we may construct a unitary transformation from $\{|l,m_l\rangle\otimes|s,m_s\rangle\}$ to $\{|j,m\rangle\}$ as

$$\mathcal{U}\left(|l,m_l\rangle\otimes|s,m_s\rangle\right) = |j,m\rangle .$$

「 " $1\otimes\frac{1}{2}=\frac{3}{2}\oplus\frac{1}{2}$ "

$: 3\times 2 = 4+2$. 」

$$\Rightarrow\ \mathcal{U} \doteq \begin{array}{c} \begin{array}{cccccc} m_l=1 & & m_l=0 & & m_l=-1 & \\ m_s=\frac12 & m_s=-\frac12 & & & & \end{array} \\ \left(\begin{array}{cc:cc:cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1/3} & \sqrt{2}/\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & \sqrt{2}/\sqrt{3} & -1/\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{3} & -\sqrt{2}/\sqrt{3} & 0 \end{array}\right) \end{array} \begin{array}{l} j\ m \\ \cdots |\frac32,\frac32\rangle \\ \cdots |\frac32,\frac12\rangle \\ \cdots |\frac32,-\frac12\rangle \\ \cdots |\frac32,-\frac32\rangle \\ \cdots |\frac12,\frac12\rangle \\ \cdots |\frac12,-\frac12\rangle \end{array}$$

" multiplet "

「 c.f.) triplet
singlet
in $\frac12\otimes\frac12$ 」

Verify:

$$\mathcal{U} J_z \mathcal{U}^\dagger = \hbar \left(\begin{array}{cccc:cc} \frac32 & & & & & \\ & \frac12 & & & & \\ & & -\frac12 & & & \\ & & & -\frac32 & & \\ \hdashline & & & & \frac12 & \\ & & & & & -\frac12 \end{array}\right).$$

* How to read out the table of CG coefficients

"just Google it! you'll find the table."

$1\times 1/2$		$3/2$		
		$+3/2$	$3/2$	$1/2$
$+1$	$+1/2$	1	$+1/2$	$+1/2$
	$+1$	$-1/2$	$1/3$	$2/3$
	0	$+1/2$	$2/3$	$-1/3$

$|\frac32,+\frac12\rangle$

$|1,1\rangle\otimes|\frac12,-\frac12\rangle$

square-root is omitted $\sqrt{2/3}$

	J M	J M	...
m_1 m_2	CG coeff.		
m_1 m_2			
⋮			

$$\overset{j\quad m}{|\tfrac32,\tfrac12\rangle} = \frac{1}{\sqrt3}\,\overset{m_1}{|1,1\rangle}\otimes\overset{m_2}{|\tfrac12,-\tfrac12\rangle} + \sqrt{\tfrac23}\,\overset{m_1}{|1,0\rangle}\otimes\overset{m_2}{|\tfrac12,\tfrac12\rangle}$$

also, $$\overset{m_1}{|1,1\rangle}\otimes\overset{m_2}{|\tfrac12,-\tfrac12\rangle} = 1/\sqrt3\,\overset{j\quad m}{|\tfrac32,\tfrac12\rangle} + \sqrt{\tfrac23}\,|\tfrac12,\tfrac12\rangle .$$

There're also computation codes to calculate CG coeffs.

in python, c/c++, ...